DEPARTMENT OF METEOROLOGY

FLORIDA STATE UNIVERSITY

TECHNICAL REPORT

Variable Vorticity Trajectories

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1. Statement of the Problem.

Thirteen years ago C.G. Rossby presented an analysis of the motion of air under conservation of its absolute vorticity which has had a considerable subsequent effect upon both theoretical and synoptic thought. From this has stemmed a more or less objective procedure for the calculation of "constant absolute vorticity trajectories", hereafter designated as CAVT. The synoptic applications have been investigated by Fultz (2) who also gave a discussion of the various assumptions entering into the theory.

One of the chief of these assumptions is that the horizontal divergence is zero. Constant vorticity trajectories are often used to predict how atmospheric flow patterns will change; however, the near-geostrophic state of the atmosphere requires a pressure change to accompany such a change in flow, and, from the continuity equation, pressure changes are associated mainly with horizontal divergence. Thus the following important question may be raised: Is the field of divergence needed in order to keep the pressure distribution in approximate balance with a changing velocity field of sufficient magnitude to in turn directly influence the flow pattern ? No attempt will be made here to answer this question directly. Instead we shall endeavor to determine the effect of a given and synoptically small divergence field upon an otherwise constant vorticity trajectory. If such a small divergence has an appreciable influence on the motion then we may be justified in concluding that the answer to this basic question is affirmative. If we find no important effect then we may be justified in assuming that such small divergences are of no great importance for the state of motion.

2. Derivation of the Equation of the Trajectory.

If one neglects solenoids, viscosity, vertical velocity, and all derivatives of vertical velocity the vorticity theorem becomes:

$$\frac{d\zeta}{dt} = -\frac{2\omega}{r}\cos\phi\, v - (2\omega\sin\phi + \zeta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\tan\phi}{r} v\right), \qquad (1)$$

where ζ is the relative vorticity about the vertical, ω is the earth's angular velocity, r the earth's radius, ϕ the latitude, u and v the horizontal velocity components directed eastward and northward along distances x and y respectively. Rossby assumed that the entire horizontal divergence vanished. We shall investigate the effect of assuming that only $\partial u/\partial x + \partial v/\partial y$ is zero, while the correction term due to the earth's curvature, $- v \tan\phi/r$, remains. The extent to which such an assumption is realistic will be discussed later; it suffices here to say that first, it is valid to prescribe such a divergence field and investigate its effects, and second, the magnitude of the divergence so assumed will be synoptically small, except near the poles. For example, at 45°N the assumed divergence will be less than 10^{-6} s^{-1} for $v = 5$ m s^{-1} whereas the synoptic divergence is of the order of 10^{-5} s^{-1}.

In order to see in advance the effect of such an assumption about the divergence let us also ignore ζ in comparison with $2\omega\sin\phi$. Equation (1) becomes:

$$\frac{d\zeta}{dt} = -\frac{2\omega}{r}\cos\phi\, v + \frac{2\omega}{r}\frac{\sin^2\phi}{\cos\phi} v.$$

The first term is the term which controls CAVT and the second term contains the additional effect introduced here. One sees that at latitude 45° the two terms are equal in magnitude and opposite in sign. Equatorward of 45° the first term dominates and the trajectories should qualitatively resemble CAVT. Poleward of 45° the second term dominates and the trajectories should curve oppositely to CAVT.

We shall now follow the derivation of CAVT by seeking steady state solutions and by assuming that vorticity expresses itself as curvature of the trajectories and not as shear.

Then:

$$\frac{d\zeta}{dt} = c\zeta' \qquad\qquad \zeta = c\psi' + \frac{\tan\phi}{r}u$$

$$v = c \sin\phi \qquad\qquad u = c\cos\phi$$

where c is the wind speed, ψ is the angle the wind makes with east (increasing counterclockwise), and a prime indicates differentiation with respect to distance along the trajectory. If we assume that c is constant and recognize that $\phi' = \sin\psi/r$, equation (1) becomes:

$$\psi'' - \frac{2\tan\phi\sin\psi}{r}\psi' - \frac{\cos\phi\sin\psi}{r^2} + \frac{2\omega}{rc}\frac{\cos 2\phi}{\cos\phi}\sin\psi = 0. \qquad (2)$$

This is the differential equation of the trajectory which for brevity may be written in the form:

$$\psi'' = \left[\alpha g(\phi) + \beta\tan\phi\,\psi' - \cos\psi\right]\frac{\sin\psi}{r^2} \qquad (3)$$

where $\alpha = -2\omega r/c$, $g(\phi) = \cos 2\phi/\cos\phi$, $\beta = 2r$.

3. Method of Solution of the Differential Equation.

Equation (3) was solved by a combined graphical-numerical method. If we are given c, ψ, ψ' and ϕ at some point along the trajectory, designated by m, then it is possible to compute the values of ψ, ψ' and ϕ at point m + 1, a small finite distance further along the trajectory.

If we replace differentials of angle and distance along the trajectory by finite differences, $\Delta\psi$ and ΔS, equation (3) may be written:

$$\Delta\psi' = \overline{\psi''}\Delta S = \left[\alpha g(\bar{\phi}) + \beta \tan\bar{\phi}\ \overline{\psi'} - \cos\bar{\psi}\right]\frac{\sin\bar{\psi}}{r^2}\Delta S. \qquad (4)$$

Here we are calculating the change in ψ' in going from m to m + 1. The bar indicates an average value of the quantity concerned in the interval from m to m + 1. The quantities $\bar{\psi}$ and $\overline{\psi'}$ may be obtained with sufficient accuracy by extrapolation of these quantities from their known values at point m. The quantity $\bar{\phi}$ may then be obtained adequately by plotting on a map an extension of the trajectory for the distance ΔS, using the angle $\bar{\psi}$, and reading the latitude at the midpoint of the extension.

Now that we have $\Delta\psi'_{m \text{ to } m+1}$ we may use the relationship

$$\psi'_{m+1} = \psi'_m + \Delta\psi'_{m \text{ to } m+1}. \qquad (5)$$

It is furthermore clearly consistent and proper to assume that

$$\Delta\psi_{m \text{ to } m+1} = \frac{\psi'_m + \psi'_{m+1}}{2}\Delta S$$

and finally that

$$\psi_{m+1} = \psi_m + \Delta\psi_{m \text{ to } m+1}. \qquad (6)$$

Thus from (5) and (6) one gets the angle and its derivative at the point m + 1, so that it is possible to continue calculating to the point m + 2.

In practice it was found useful to choose ΔS = 1 deg. of latitude. This is small enough to make the finite difference approximation a good one, and large enough to keep the number of calculations per trajectory within reasonable limits.

4. Resultant Trajectories.

Figs. 1-5 give examples of some of the trajectories followed by air moving under the assumptions made here.

In fig. 1 we have examples of trajectories beginning from an inflection point at 45°N. Since at this latitude the two factors affecting $d\zeta/dt$ cancel, the air does not change curvature appreciably until it moves far away from 45°. To the south thereof it curves cyclonically, as do CAVT, but more slowly because of the cancelling effect of the divergence term. When the air goes north of 45° the divergence term predominates and continues to turn the trajectory cyclonically. This produces the closed loop shown. Thus one of the salient characteristics of these paths is that south of 45° latitude westerly trajectories are sinusoidal in nature and may be described as "stable". Poleward of 45° westerly trajectories break up into closed loops and are, in a certain sense, "unstable". A comparison to the equivalent CAVT reveals that the present trajectories are of much greater wave length. An increase in speed causes an increase in the dimensions of the trajectory.

Fig. 2 shows the result of changing the initial wind direction to due north. The northern "loop" becomes much larger. Again this trajectory differs appreciably from the CAVT.

In fig. 3 the behavior of initially easterly winds poleward of 45° is seen. Just as westerlies give sinusoidal paths in CAVT so in this theory easterlies at latitudes such as this give "stable oscillations." The CAVT for easterlies are composed of closed loops and are "unstable".

Fig. 4 shows a complex trajectory beginning as a south wind at 60°N. At first the effect of convergence in poleward moving air predominates and the parcel turns cyclonically. Then as the air is turned southward

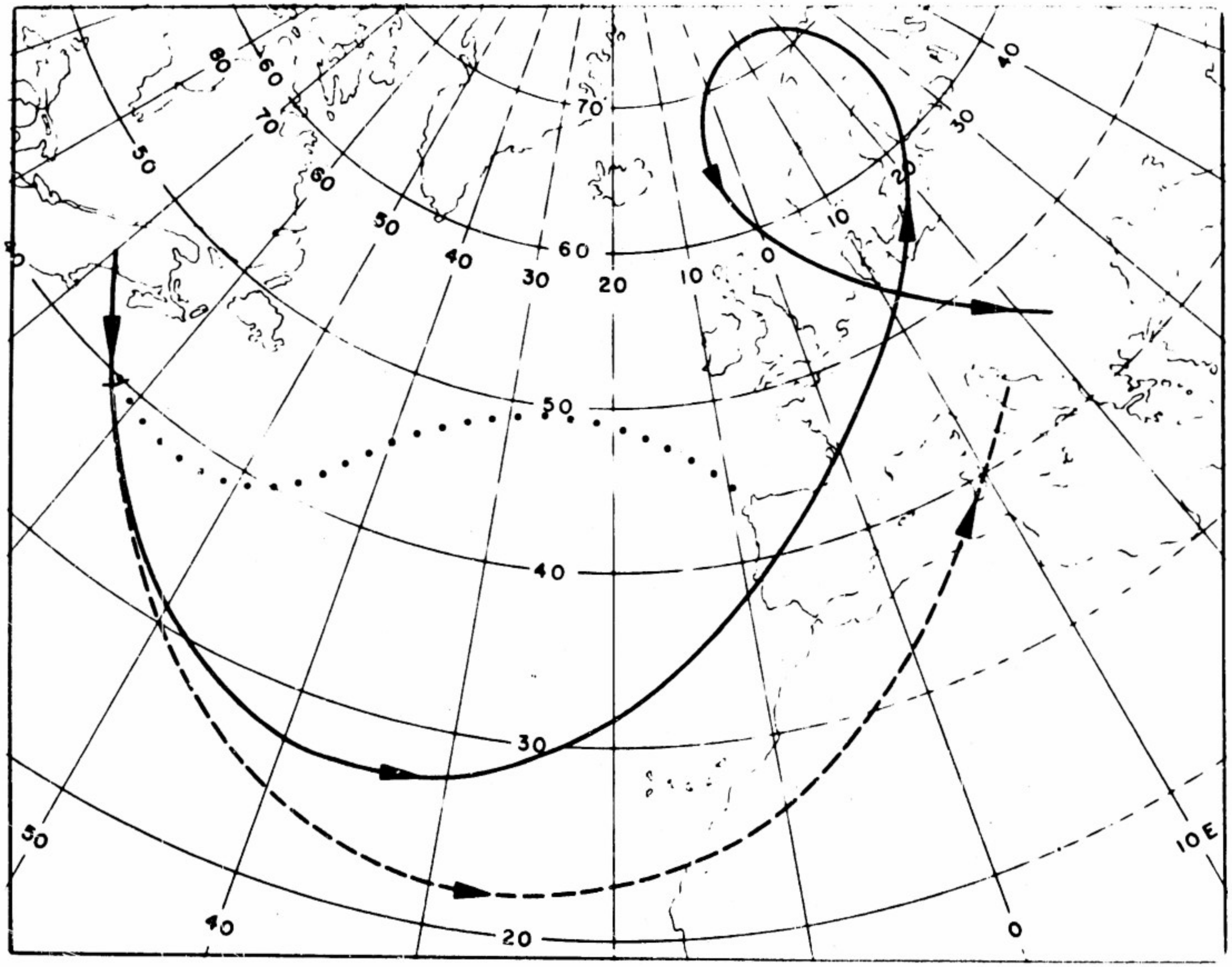

FIG I

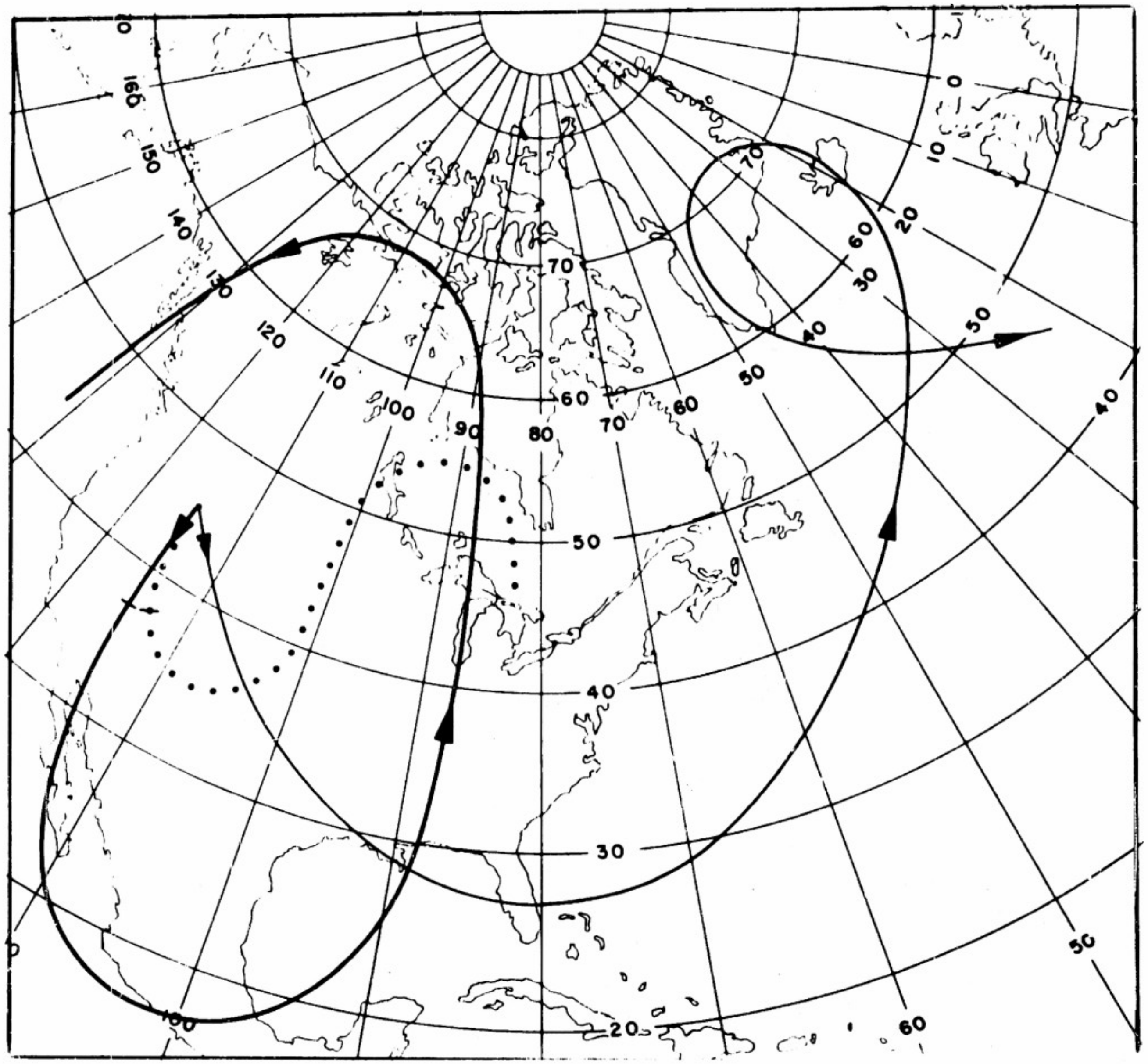

FIG 2

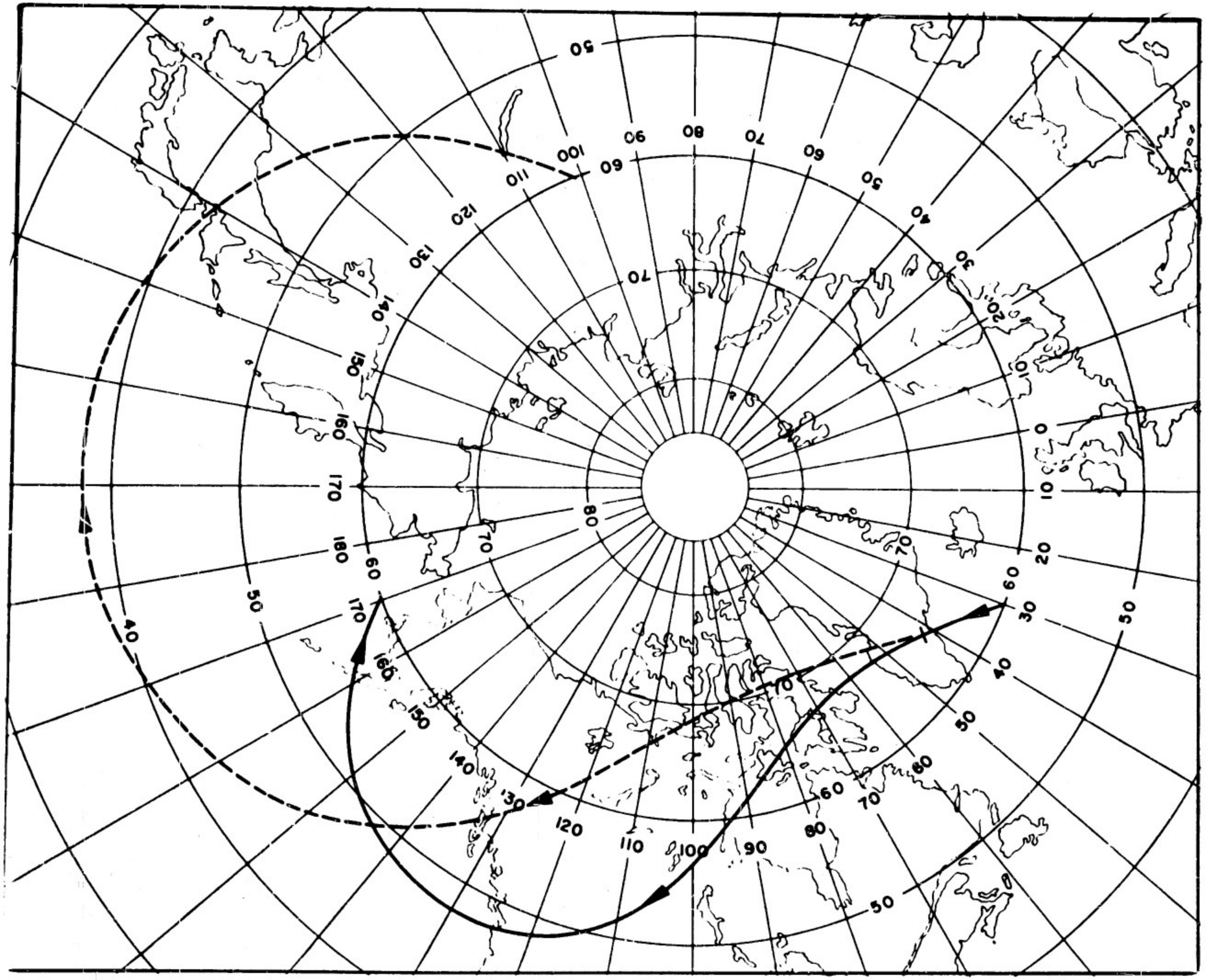

FIG 3

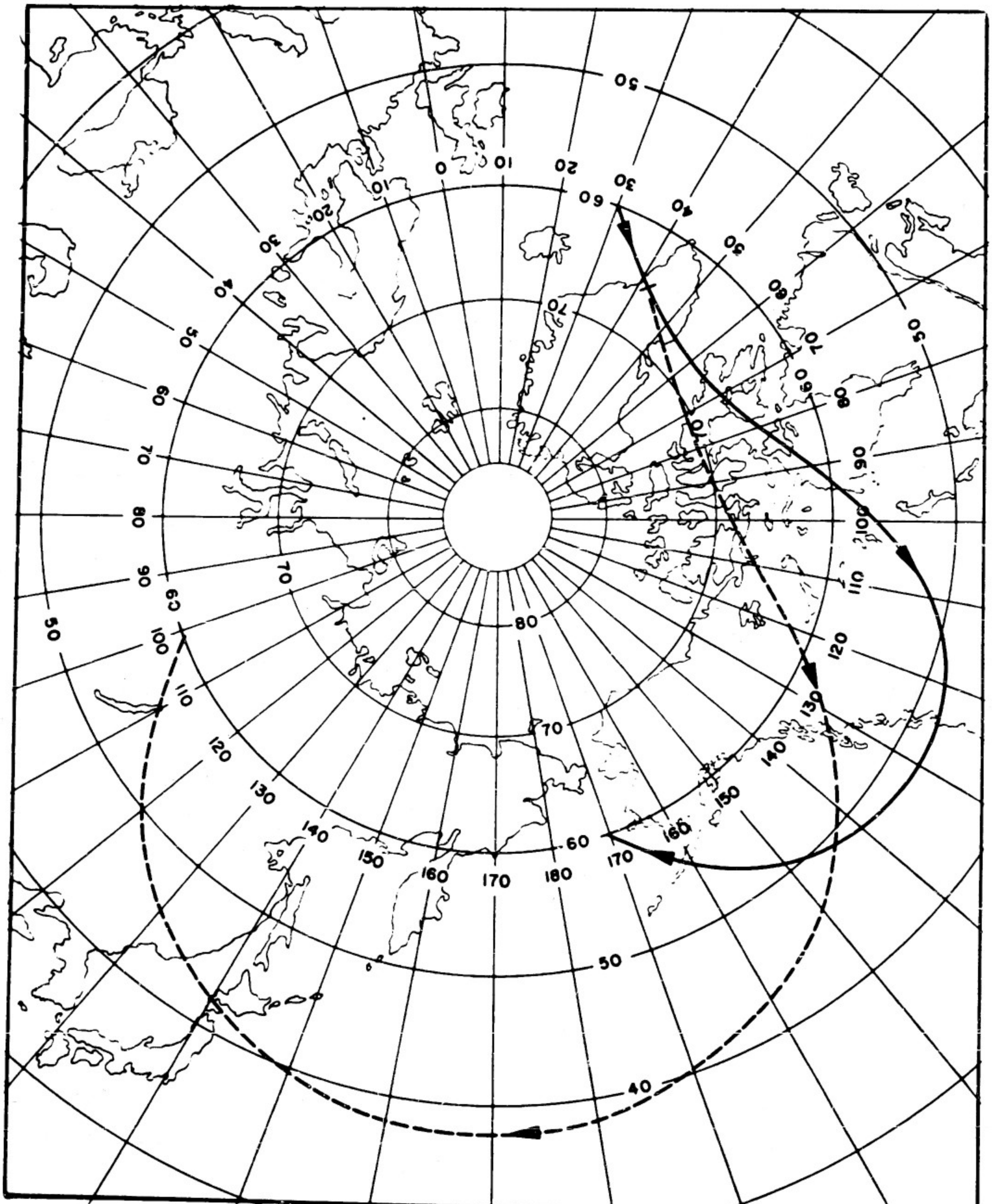

FIG 3

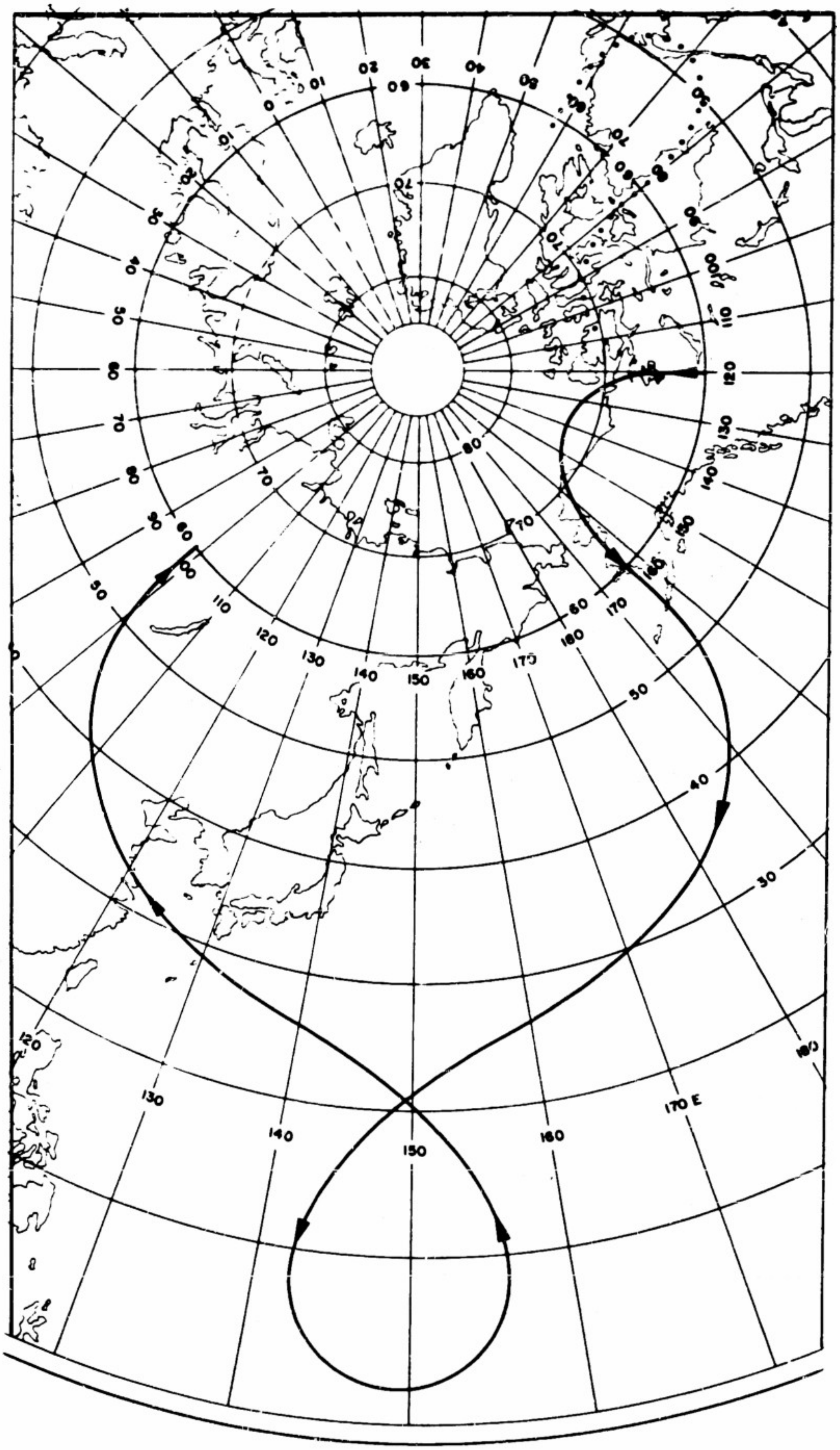

FIG 4

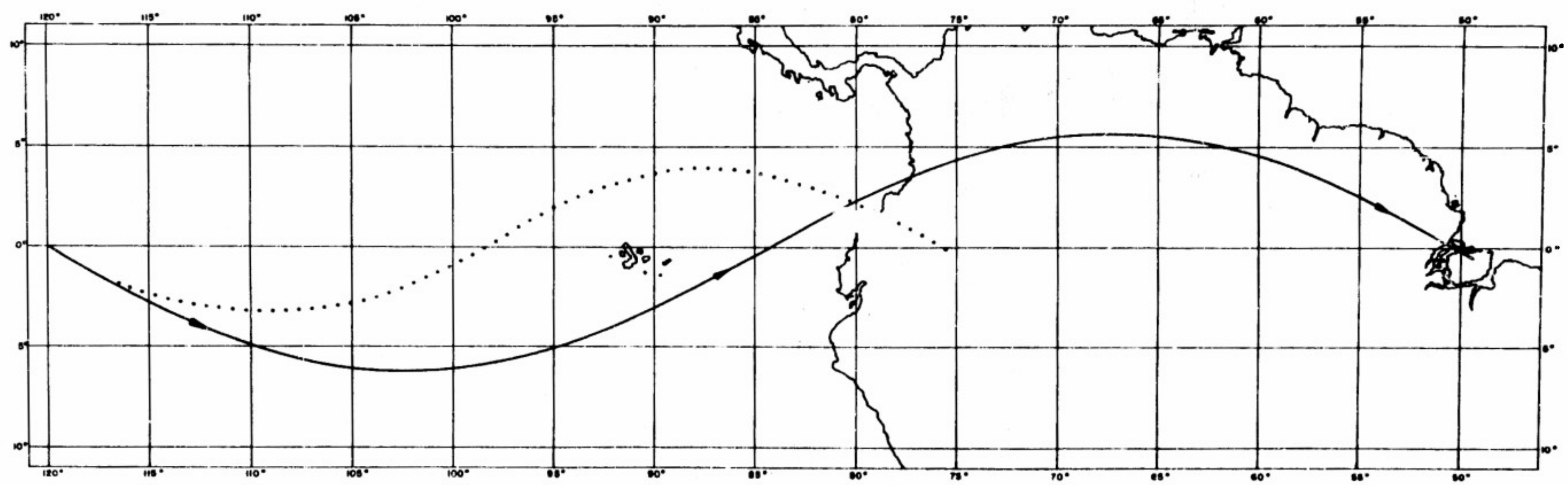

FIG 5

divergence begins and anticyclonic vorticity is generated. However, in the course of this motion the air has gone south of 45° so that the non-divergent latitude term becomes dominant. Thus, as in CAVT, the southward moving air builds up cyclonic vorticity which turns it through a closed loop in southerly latitudes.

Finally, there is an example of an equatorial trajectory in fig. 5. At these latitudes the added divergence term in equation (1) should be negligibly small since it has tan ϕ as a factor. Specifically, in the trajectory shown the added term is less than one percent of the Rossby latitude term. Nevertheless, the figure shows a considerable difference in wavelength between this and the corresponding CAVT. It has been verified that this difference cannot be attributed to approximation in the computing procedure. There seems to be no escape from the conclusion that even so small a term may, if consistent in its effect, have an appreciable influence on the result. There may be a relationship between this case and what Birkhoff (1) has called "asymptotic parodoxes" in differential equations.

5. Significance of the Results

It appears from these trajectories that a divergence field of relatively small magnitude may have an important effect upon the air trajectory. This lends support to the suspicion that the divergence associated with moving pressure systems has a direct and significant effect upon the motion of the air, and casts doubt upon the extent to which the results of non-divergent theories are quantitatively applicable to the real atmosphere.

One may now inquire how realistic are the divergence fields postulated here and, therefore, how applicable are these trajectories to the real atmosphere. One must recognize that there is a certain degree of artificiality in any arbitrary specification about the divergence. The artificiality in the present specification is seen most clearly if one considers air near the pole where the divergence assumed here approaches infinity. This is clearly inappropriate and it must be true that $\partial u/\partial x + \partial v/\partial y$ also approaches infinity near the poles, but with opposite sign from the latitude term, so that the total divergence remains finite. With this in mind, nevertheless certain features of the present assumption are quite realistic provided one stays away from high latitudes. First, we are imposing convergence for southerly winds and divergence for northerly winds (in the Northern Hemisphere). This corresponds qualitatively to the synoptic observation of a tendency for convergence ahead of troughs aloft and divergence ahead of ridges aloft. Second, the divergence postulated here, $- v \tan \phi/r$, is the divergence of the geostrophic wind. That is, <u>insofar as the wind field strives to remain geostrophic it imposes upon itself the divergence field investigated here</u>. Since the quasi-geostrophic nature of the wind in middle latitudes is well known, it seems likely that the divergence under discussion is one of the basic components of the total divergence field.

When one examines the theoretical trajectories themselves one sees certain features which are realistic and others which are not. The tendency shown in fig. 1 for westerlies south of 45° to be "stable", but north of 45° to break up into closed patterns, is similar to the situation on normal upper air charts. This normal feature is not explained by a non-divergent, barotropic theory such as that for CAVT. On the other hand, for instance, the trajectory shwon in fig. 4 is rather complex and is not such as are normally observed. This may be because this trajectory initially comes close to the pole where other divergent effects must be important, and because it later goes to very low latitudes where the tendency towards geostrophic balance is quite weak.

The claim is not advanced that these trajectories are a significant practical improvement over CAVT, nor that they are useful as prognostic tools. They do seem to point to the great importance of the direct effect of divergence upon the general flow via the vorticity equation, and there are certain aspects of the results which carry versimilitude. The results also illustrate that fact that one must be very careful in ignoring small terms in differential equations, since they may have large effects.

BIBLIOGRAPHY

1. Birkhoff, G., 1950: Hydrodynamics, Princeton, Princeton University Press. p4.

2. Fultz, D., 1945: Upper Air trajectories and weather forecasting. Misc. report No. 19, U. of Chicago.

3. Rossby, C.-G., 1940: Planetary flow patterns in the atmosphere. Quart. J. r. meteor. Soc., 66, Suppl. 68.